Are These the Most Beautiful?

David Wells

In the Fall 1988 *Mathematical Intelligencer* (vol. 10, no. 4) readers were asked to evaluate 24 theorems, on a scale from 0 to 10, for beauty. I received 76 completed questionnaires, including 11 from a preliminary version (plus 10 extra, noted below.)

One person assigned each theorem a score of 0, with the comment, "Maths is a tool. Art has beauty"; that response was excluded from the averages listed below, as was another that awarded very many zeros, four who left many blanks, and two who awarded numerous 10s.

The 24 theorems are listed below, ordered by their average score from the remaining 68 responses.

Rank	Theorem	Average
(1)	$e^{i\pi} = -1$	7.7
(2)	Euler's formula for a polyhedron: $V + F = E + 2$	7.5
(3)	The number of primes is infinite.	7.5
(4)	There are 5 regular polyhedra.	7.0
(5)	$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \pi^2/6.$	7.0
(6)	A continuous mapping of the closed unit disk into itself has a fixed point.	6.8
(7)	There is no rational number whose square is 2.	6.7
(8)	π is transcendental.	6.5
(9)	Every plane map can be coloured with 4 colours.	6.2
(10)	Every prime number of the form $4n + 1$ is the sum of two integral squares in exactly one way.	6.0
(11)	The order of a subgroup divides the order of the group.	5.3
(12)	Any square matrix satisfies its characteristic equation.	5.2
(13)	A regular icosahedron inscribed in a regular octahedron divides the edges in the Golden Ratio.	5.0
(14)	$\frac{1}{2 \times 3 \times 4} - \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} - \ldots = \frac{\pi - 3}{4}.$	4.8
(15)	If the points of the plane are each coloured red, yellow, or blue,	4.7

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David Wells won a scholarship to Cambridge University, England, but then failed his degree, a rare achievement. Abandoning plans to become a professional mathematician, he became a school teacher, then a puzzle composer and game inventor (having once been British under-21 chess champion). More recently he has been a freelance author and lecturer, concentrating on mathematical education.

	there is a pair of points of the same colour of mutual distance unity.	
(16)	The number of partitions of an integer into odd integers is equal to the number of partitions into distinct integers.	4.7
(17)	Every number greater than 77 is the sum of integers, the sum of whose reciprocals is 1.	4.7
(18)	The number of representations of an odd number as the sum of 4 squares is 8 times the sum of its divisors; of an even number, 24 times the sum of its odd divisors.	4.7
(19)	There is no equilateral triangle whose vertices are plane lattice points.	4.7
(20)	At any party, there is a pair of people who have the same number of friends present.	4.7
(21)	Write down the multiples of root 2, ignoring fractional parts, and underneath write the numbers missing from the first sequence. 1 2 4 5 7 8 9 11 12 3 6 10 13 17 20 23 27 30 The difference is $2n$ in the nth place.	4.2
(22)	The word problem for groups is unsolvable.	4.1
(23)	The maximum area of a quadrilateral with sides a,b,c,d is $[(s-a)(s-b)(s-c)(s-d)]^{1/2}$, where s is half the perimeter.	3.9
(24)	$\frac{5[(1-x^5)(1-x^{10})(1-x^{15})\ldots]^5}{[(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots]^6} = p(4) + p(9)x + p(14)x^2 + \ldots,$ where $p(n)$ is the number of partitions of n.	3.9

The following comments are divided into themes. Unattributed quotes are from respondents.

Theme 1: Are Theorems Beautiful?

Tony Gardiner argued that "Theorems aren't usually 'beautiful'. It's the ideas and *proofs* that appeal," and remarked of the theorems he had not scored, "The rest are hard to score—either because they aren't really beautiful, however important, or because the formulation given gets in the way. . . ." Several respondents disliked judging theorems. (How many readers did not reply for such reasons?)

Benno Artmann wrote "for me it is impossible to judge a 'pure fact' "; this is consistent with his interest in Bourbaki and the axiomatic development of structures.

Thomas Drucker: "One does not have to be a Russellian to feel that much of mathematics has to do with deriving consequences from assumptions. As a result, any 'theorem' cannot be isolated from the assumptions under which it is derived."

Gerhard Domanski: "Sometimes I find a problem more beautiful than its solution. I find also beauty in mathematical ideas or constructions, such as the Turing machine, fractals, twistors, and so on. . . . The ordering of a whole field, like the work of Bourbaki . . . is of great beauty to me."

R. P. Lewis writes, '(1) . . . I award 10 points not so much for the equation itself as for Complex Analysis as a whole.' To what extent was the good score for (4) a vote for the beauty of the Platonic solids themselves?

Theme 2: Social Factors

Might some votes have gone to (1), (3), (5), (7), and (8) because they are 'known' to be beautiful? I am suspicious that (1) received so many scores in the 7–10 range. This would surprise me, because I suspect that mathematicians are more independent than most people [13] of others' opinions. (The ten extra forms referred to above came from Eliot Jacobson's students in his number theory course that emphasises the role of beauty. I noted that they gave no zeros at all.)

Theme 3: Changes in Appreciation over Time

There was a notable number of low scores for the high rank theorems. Le Lionnais has one explanation [7]: "Euler's formula $e^{i\pi} = -1$ establishes what appeared in its time to be a fantastic connection between the most important numbers in mathematics . . . It was generally considered 'the most beautiful formula of mathematics' . . . Today the intrinsic reason for this compatibility has become so obvious that the same formula now seems, if not insipid, at least entirely natural." Le Lionnais, unfortunately, does not qualify "now seems" by asking, "to whom?"

How does judgment change with time? Burnside [1], referring to "a group which is . . . abstractly equivalent to that of the permutations of four symbols," wrote, "in the latter form the problem presented would to many minds be almost repulsive in its naked formality . . ."

Earlier [2], perspective projection was, "a process occasionally resorted to by geometers of our own country, but generally esteemed . . . to be a species of